

Exact three dimensional black hole with gauge fields in string theory

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Abstract

We have obtained exact three dimensional BTZ type solutions with gauge fields, for string theory on a gauge symmetric gravitational background constructed from semi-simple extension of the Poincaré algebra (and the Maxwell algebra) in $2 + 1$ dimensions. We have studied the models for two non-Abelian and Abelian gauge fields solutions and shown that the related sigma models for each of these backgrounds is a $SL(2, R)$ WZW (Wess-Zumino-Witten) model and that they are classically canonically equivalent. We have also obtained the dual solution for the Abelian case and by interpreting the new field strength tensors of the Abelian solution as electromagnetic field strength tensors shown that dual models coincide with the charged black string solution.

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1 Introduction

In order to overcome some complexities of four dimensional gravity, many of researchers have studied the gravity models in lower dimensions. For instance, they hope that some properties of lower dimensional black holes help them to model those of the four dimensional black holes. One of such attempts has resulted in the construction of the three dimensional rotating BTZ black hole [1]. Nearly two decade ago, Witten [2] has shown that an exact two dimensional black hole in string theory could be obtained by gauging a one dimensional subgroup $U(1)$ of $SL(2, R)$. Also, exact three dimensional black string [3] and black hole [4] solutions in string theory have been obtained. In other attempt, the $SL(2, R)$ WZW model and it's relation to string theory in AdS_3 has been studied in details in three different papers [5, 6, 7]. In those papers, the structure of the Hilbert space of the WZW model and the spectrum of physical states of the string theory have been determined [5], and also the one loop amplitude [6] and the correlation functions of the model [7] have been studied.

Recently, the Maxwell [8, 9] and Semi-simple extension of Poincaré symmetries was applied to construct gauge symmetric gravity models in $3 + 1$ [10, 11] and $2 + 1$ [12] dimensions (see also [13, 14]). Here, we apply these symmetries to obtain an exact three dimensional black hole with gauge fields in string theory by introducing a new extended anti-symmetric B-field and try to obtain some solutions for equations of motion of the low energy string effective action [15, 16]. The outlines of the paper is as follows:

In section two, we construct a low energy string effective action in $2 + 1$ dimensions by use of new gauge field strengths and obtain it's equations of motion. In section three, we solve the equations of motion using the *BTZ* metric and obtain two different non-Abelian and Abelian solutions (Abelian solution has no contribution of interaction terms in new gauge field strengths). Then, we show that both solutions are *exact solutions* whereas the sigma models for each of these backgrounds is a $SL(2, R)$ Wess-Zumino-Witten model. We also show that two sigma models corresponding to the two non-Abelian and Abelian solutions are classically canonically equivalent. We interpret the new gauge field strengths of the Abelian solution as an electromagnetic field strength tensors, and obtain the corresponding three different electric and magnetic fields. In section four, using duality transformation, we calculate dual solutions of the Abelian solution with respect to both spacelike (φ) and timelike coordinate isometry symmetries and show that the dual solutions correspond to the charged black string solution. By obtaining both φ -dual and t -dual electric and magnetic fields, we show that duality relates the electric fields to the dual magnetic fields and vice versa. We present some concluding remarks in section five. In appendix, we cast both the Maxwell algebra and the semi-simple extension of the Poincaré algebra (Ads-Lorentz algebra¹) in $2 + 1$ dimensional spacetime from the anti-de Sitter (Ads) algebra ($so(2,2)$ in $2 + 1$ dimensions) by use of the S-expansion procedure with the appropriate semigroups S and \bar{S} [17, 18].

2 Gauge fields in string action from semi-simple extension of the Poincaré algebra in $2 + 1$ dimensions

Let us consider the semi-simple extension of the Poincaré algebra (or the Maxwell algebra for $\lambda = 0$) with the basis $X_B = \{P_a, J_a, Z_a\}$ in $2 + 1$ dimensions [12] as follows:²

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J^c, & [J_a, P_b] &= \epsilon_{abc} P^c, & [P_a, P_b] &= k \epsilon_{abc} Z^c, \\ [J_a, Z_b] &= \epsilon_{abc} Z^c, & [P_a, Z_b] &= -\frac{\lambda}{k} \epsilon_{abc} P^c, & [Z_a, Z_b] &= -\frac{\lambda}{k} \epsilon_{abc} Z^c, \end{aligned} \quad (1)$$

where k and λ are constants, and Z_a are new generators which are added to ordinary Poincaré generators P_a and J_a to extend the algebra. One can construct gauge fields h_μ which are Lie algebra valued one form $h = h_\mu dx^\mu$ as follows:

$$h_\mu(x) = h_\mu^B(x) X_B = e_\mu^a(x) P_a + \omega_\mu^a(x) J_a + A_\mu^a(x) Z_a, \quad (2)$$

where x^μ ($\mu = 0, 1, 2$) and $a, b = 0, 1, 2$ are spacetime coordinates and Lie algebra indices, respectively. Using these gauge fields $(e_\mu^a(x), \omega_\mu^a(x), A_\mu^a(x))$, one can write the new field strengths $F_{\mu\nu}^a$ as follows [12]:

$$F_{\mu\nu}^a = \partial_{[\mu} A_{\nu]}^a + \epsilon_{bc}^a (k e_\mu^b e_\nu^c + \omega_\mu^b A_\nu^c + A_\mu^b \omega_\nu^c) - \frac{\lambda}{k} \epsilon_{bc}^a A_\mu^b A_\nu^c. \quad (3)$$

¹The semi-simple extension of the Poincaré algebra is the direct sum of the *Ads* algebra and the Lorentz algebra, i.e. $so(2,2) \oplus so(2,1)$ in $2 + 1$ dimensional spacetime, and then is called the Ads-Lorentz algebra.

²The relationship between Maxwell algebra and semi-simple extension of the Poincaré algebra with the *Ads* algebra has been discussed in the appendix.

where the gauge fields e_μ^a , ω_μ^a and A_μ^a are vierbein, spin connection and new non-Abelian gauge fields³ (corresponding to the new generators Z_a) respectively. Now we consider the low energy string effective action in 2 + 1 dimensional spacetime \mathcal{M} by use of these gauge fields as follows:⁴ [4, 15]

$$S = \int_{\mathcal{M}} d^3x \sqrt{-g} e^{-2\phi} \left[R + \frac{4}{K} + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{12} H'_{\mu\nu\rho}{}^a H'^{\mu\nu\rho}{}_a \right] \quad (4)$$

where R is the Ricci scalar of \mathcal{M} and ϕ is the dilaton field, and furthermore $H_{\mu\nu\rho}$ and $H'_{\mu\nu\rho}{}^a$ ⁵ are defined as follows:

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad (5)$$

$$H'_{\mu\nu\rho}{}^a = \partial_\mu F_{\nu\rho}{}^a + \partial_\nu F_{\rho\mu}{}^a + \partial_\rho F_{\mu\nu}{}^a, \quad (6)$$

so that $B_{\mu\nu}$ is an antisymmetric field and $F_{\mu\nu}{}^a$ are the new antisymmetric field strengths. Now, one can find the following equations of motion (the beta functions) by variations of the above action with respect to $g^{\mu\nu}$, $B^{\nu\rho}$, $F^{\nu\rho}{}_a$ and ϕ respectively,

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - \frac{1}{8} H'_{\mu\rho\sigma}{}^a H_\nu{}^{\rho\sigma}{}_a &= 0, \\ \nabla^\mu (e^{-2\phi} H_{\mu\nu\rho}) &= 0, \\ \nabla^\mu (e^{-2\phi} H'_{\mu\nu\rho}{}^a) &= 0, \end{aligned} \quad (7)$$

$$4\nabla^2 \phi - 4(\nabla\phi)^2 + R + \frac{4}{K} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{12} H'_{\mu\nu\rho}{}^a H'^{\mu\nu\rho}{}_a = 0,$$

where these equations are zeros of the beta functions (at one loop) $\beta(G)$, $\beta(B)$, $\beta(F^a)$ and $\beta(\phi)$ (respectively) for the following sigma model [16]

$$I = \int_{\Sigma} d^2\sigma \sqrt{g} \left(G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\beta} + B'_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} + \frac{1}{2} R^{(2)} \phi(X) \right), \quad (8)$$

where Σ is the worldsheet, $R^{(2)}$ is the scalar curvature of the worldsheet metric $g_{\alpha\beta}$, and $\epsilon^{\alpha\beta}$ is an anti-symmetric 2-tensor, normalized so that $\sqrt{g}\epsilon^{12} = +1$, and the extended antisymmetric B-field has the following form:

$$B'_{\mu\nu}(X) = B_{\mu\nu}(X) + F_{\mu\nu}^0(X) + F_{\mu\nu}^1(X) + F_{\mu\nu}^2(X) = B_{\mu\nu}(X) + F_{\mu\nu}^a(X) \xi_a, \quad (9)$$

such that we have $\xi_a = (1, 1, 1)$.

3 Black hole solutions

In this section, we will try to obtain a solution for the equations (7). We know that the 2 + 1 dimensional Einstein-Hilbert action with cosmological constant term

$$S = \int d^3x \sqrt{-g} (R - 2\Lambda), \quad (10)$$

has a BTZ black hole solution as follows: [1]

$$ds^2 = -N^2(r) dt^2 + \frac{1}{N^2(r)} dr^2 + r^2 (N^\phi(r) dt + d\phi)^2, \quad (11)$$

³Note that for the general form of the field strength (3) the gauge fields A_μ^a are non-Abelian. For the Abelian case we have:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a,$$

i.e. for these gauge fields A_μ^a we have:

$$\epsilon_{bc}^a (k e_\mu^b e_\nu^c + \omega_\mu^b A_\nu^c + A_\mu^b \omega_\nu^c) - \frac{\lambda}{k} \epsilon_{bc}^a A_\mu^b A_\nu^c = 0.$$

⁴Note that the above action is the ordinary string effective action in 2 + 1 dimensions where the last term is added.

⁵The algebra indices a can be taken up and down by the ad-invariant metric Ω_{ab} of the algebra [12].

with

$$N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \quad N^\phi(r) = -\frac{J}{2r^2}, \quad (12)$$

where $\Lambda = -\frac{1}{\ell^2}$ is the negative cosmological constant, furthermore M and J are the mass and angular momentum of the black hole, respectively. Here $[x^0, x^1, x^2] = [t, r, \varphi]$ are the coordinates of the spacetime. We use the above BTZ metric to solve the equations of motion (7) assuming that all fields are a function of the radial coordinate only. Here, we analyse two interesting non-Abelian and Abelian gauge field solutions.

3.1 Non-Abelian case

Using the form of the metric as (11) and (12) the equations of motion (7) have a solution as follows:

$$\begin{aligned} B_{20}(r) &= \frac{r^2}{\ell}, & \phi(r) &= 0, & K &= \ell^2, \\ \omega^0(r) &= \frac{v(r)}{D_2 r^2 + D_1} \left(\pm r^2 \sqrt{D_2^2 - D_4^2} - krN(r) + D_5 \right) dt + u(r)dr, \\ \omega^1(r) &= w(r)dr, & \omega^2(r) &= v(r)dt + z(r)dr, \\ A^0(r) &= \frac{D_4 r^2 + D_3}{v(r)} \left(-\frac{J}{2r^2} dt + d\varphi \right) + y(r)dr, \\ A^1(r) &= \frac{D_2 r^2 + D_1}{v(r)} \left(-\frac{J}{2r^2} dt + d\varphi \right) + q(r)dr, & A^2(r) &= s(r)dr, \end{aligned} \quad (13)$$

where D_1, D_2, D_3, D_4, D_5 are arbitrary constants and $v(r) \neq 0, u(r), w(r), y(r), q(r), s(r), z(r)$ are arbitrary functions of radial coordinate r only. Now, using (3), we find the following nonzero components of the new antisymmetric fields (3) for the non-Abelian gauge fields A_μ^a :

$$\begin{aligned} F_{20}^0(r) &= D_2 r^2 + D_1, \\ F_{20}^1(r) &= D_4 r^2 + D_3, \\ F_{20}^2(r) &= r^2 \sqrt{D_2^2 - D_4^2} + D_5, \end{aligned} \quad (14)$$

which yields the following extended antisymmetric B-field:

$$\begin{aligned} B'_{20}(r) &= B_{20}(r) + F_{20}^0(r) + F_{20}^1(r) + F_{20}^2(r) \\ &= \left(\frac{1}{\ell} + D_2 + D_4 + \sqrt{D_2^2 - D_4^2} \right) r^2 + D_1 + D_3 + D_5. \end{aligned} \quad (15)$$

Although this is a solution for the one-loop beta function equations, one can show that by selecting $D_2 = D_4 = -\frac{1}{\ell}$, this solution is also an *exact solution* of the beta function equations in all loops. Indeed, the sigma model (8) with this background $(G_{\mu\nu}, B'_{\mu\nu})$ is a $SL(2, R)$ Wess-Zumino-Witten model. One can easily check this, by using the following $SL(2, R)$ group element:

$$g(t, r, \varphi) = \begin{pmatrix} \hat{r} e^{-\hat{\varphi}} & \sqrt{\ell^2 - \hat{r}^2} e^{\frac{\hat{t}}{\ell}} \\ \sqrt{\ell^2 - \hat{r}^2} e^{-\frac{\hat{t}}{\ell}} & -\hat{r} e^{\hat{\varphi}} \end{pmatrix}, \quad (16)$$

in the WZW action

$$S_{WZW} = \frac{\bar{k}}{4\pi} \int d^2z \operatorname{Tr}(g^{-1} \partial g g^{-1} \bar{\partial} g) - \frac{\bar{k}}{12\pi} \int \operatorname{Tr}(g^{-1} dg)^3,$$

where

$$\hat{r} = \ell \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}, \quad \hat{t} = \frac{r_+}{\ell} t + r_- \varphi, \quad \hat{\varphi} = \frac{r_-}{\ell^2} t + \frac{r_+}{\ell} \varphi, \quad (17)$$

so that \bar{k} is level of the WZW model, and the horizons $r = r_\pm$ of the black hole have the following relation to it's mass and angular momentum (M, J)

$$r_\pm = \frac{\ell}{2} \left(\sqrt{M + \frac{J}{\ell}} \pm \sqrt{M - \frac{J}{\ell}} \right). \quad (18)$$

3.2 Abelian (electromagnetic) case

Using the form of the metric as (11) and (12), we find another solution for the equations of motion (7) as follows:

$$\begin{aligned}
B_{20}(r) &= \frac{r^2}{\ell}, & \phi(r) &= 0, & K &= \ell^2, \\
\omega^0(r) &= \frac{g(r)}{r} N(r) dt + \left(\frac{-f(r)h(r)g(r) + p(r)(\lambda r^2 + (g(r))^2)}{kr^2} \right) dr, \\
\omega^1(r) &= -\frac{J}{2r^2} g(r) dt + f(r) dr + g(r) d\varphi, & \omega^2(r) &= \frac{-\lambda r}{g(r)N(r)} dr, \\
A^0(r) &= -\frac{J}{2r^2} h(r) dt + p(r) dr + h(r) d\varphi, \\
A^1(r) &= \frac{Jk}{2g(r)} dt - \frac{kr^2}{g(r)} d\varphi, & A^2(r) &= \frac{-kr}{g(r)N(r)} dr,
\end{aligned} \tag{19}$$

where $g(r) \neq 0, f(r), h(r), p(r)$ are arbitrary functions of radial coordinate only. Note that for this solution, A_μ^a is Abelian and hence yields zero contributions for the coupling terms in all components of the new field strengths $F_{\mu\nu}^a$ (i.e. we have $\epsilon_{bc}^a (k e_\mu^b e_\nu^c + \omega_\mu^b \omega_\nu^c + A_\mu^b \omega_\nu^c) - \frac{\lambda}{k} \epsilon_{bc}^a A_\mu^b A_\nu^c = 0$), and therefore $F_{\mu\nu}^a$ can be interpreted as electromagnetic field strength tensors without coupling terms as follows:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a. \tag{20}$$

For the above solution, all of the nonzero components of the electromagnetic field strength tensors are as follows:

$$\begin{aligned}
F_{01}^0(r) &= -\frac{J}{r^3} h(r) + \frac{J}{2r^2} \frac{d}{dr} h(r), & F_{21}^0(r) &= -\frac{d}{dr} h(r), \\
F_{01}^1(r) &= \frac{Jk}{2g^2(r)} \frac{d}{dr} g(r), & F_{21}^1(r) &= \frac{2kr}{g(r)} - \frac{kr^2}{g^2(r)} \frac{d}{dr} g(r).
\end{aligned} \tag{21}$$

In this way, the extended B-field components have the following forms:

$$\begin{aligned}
B'_{20}(r) &= B_{20}(r) = \frac{r^2}{\ell}, \\
B'_{01}(r) &= B_{01}(r) + F_{01}^0(r) + F_{01}^1(r) + F_{01}^2(r) = -\frac{J}{r^3} h(r) + \frac{J}{2r^2} \frac{d}{dr} h(r) + \frac{Jk}{2g^2(r)} \frac{d}{dr} g(r), \\
B'_{21}(r) &= B_{21}(r) + F_{21}^0(r) + F_{21}^1(r) + F_{21}^2(r) = -\frac{d}{dr} h(r) + \frac{2kr}{g(r)} - \frac{kr^2}{g^2(r)} \frac{d}{dr} g(r).
\end{aligned} \tag{22}$$

where we assume that $B_{01}(r) = B_{21}(r) = 0$. This solution is also an exact solution. One can show that using the $SL(2, R)$ group element (16) with

$$\hat{r} = \ell \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}, \quad \hat{t} = \frac{r_+}{\ell} t - r_- \varphi, \quad \hat{\varphi} = \frac{r_-}{\ell^2} t - \frac{r_+}{\ell} \varphi, \tag{23}$$

the sigma model (8) with the background $(G_{\mu\nu}, B'_{\mu\nu})$ is a $SL(2, R)$ WZW model. Now, we show that two sigma models corresponding to the two discussed non-Abelian (13) and Abelian (19) solutions are classically canonically equivalent. Indeed, by assuming the following relations among the arbitrary functions in (13) and (19),

$$\begin{aligned}
v(r) &= \frac{2r^2 \sqrt{-\Lambda}}{h(r) - \frac{kr^2}{g(r)} - \int dr \frac{kr}{N(r)} + D}, \\
y(r) &= \left(\frac{Jk}{rg(r)} - \frac{Jh(r)}{r^3} \right) \frac{1}{v(r)} + \frac{2J\sqrt{-\Lambda}}{rv^2(r)} - q(r) + \frac{kN(r)}{r\sqrt{-\Lambda}} (s(r) - q(r)),
\end{aligned} \tag{24}$$

$$u(r) = w(r) - 2z(r) + \frac{\lambda}{k} \left(y(r) - q(r) + 2s(r) \right),$$

then, these two sigma models can be canonically related to each other by the following relations [19]

$$\begin{aligned} (C^{-1})^{\mu\rho}(G_{\rho\lambda} + B'_{\rho\lambda} + A_{\rho\lambda})(C^{-1})^{\kappa\lambda}(\tilde{G}_{\kappa\nu} + \tilde{B}'_{\kappa\nu} + \tilde{A}_{\kappa\nu}) &= \delta^\mu_\nu, \\ (G_{\mu\nu} + B'_{\mu\nu}) &= (G_{\mu\rho} + B'_{\mu\rho} + A_{\mu\rho})(C^{-1})^{\lambda\rho}(\tilde{G}_{\lambda\nu} + \tilde{B}'_{\lambda\nu}), \end{aligned} \quad (25)$$

where $(G_{\mu\nu}, B'_{\mu\nu})$ and $(\tilde{G}_{\mu\nu}, \tilde{B}'_{\mu\nu})$ are the non-Abelian (13) and the Abelian (19) solutions respectively, such that $C_{\mu\nu}, A_{\mu\nu}$ and $\tilde{A}_{\mu\nu}$ have the following forms

$$C_{\mu\nu} = G_{\mu\nu} + B'_{\mu\nu}, \quad A_{\mu\nu} = \tilde{A}_{\mu\nu} = 0. \quad (26)$$

Now, for the Abelian solution, the electric and magnetic fields can be written in terms of electromagnetic field strength tensors as follows:

$$\begin{aligned} E_r^{(a)} &= -F^{01a} = -(g^{00}g^{11}F_{01}^a + g^{02}g^{11}F_{21}^a) = F_{01}^a - N^\phi(r)F_{21}^a, \quad E_\phi^{(a)} = 0, \\ B_z^{(a)} &= rF^{12a} = -r(g^{11}g^{20}F_{01}^a + g^{11}g^{22}F_{21}^a) = \frac{J}{2r}F_{01}^a - \left(\frac{N^2 - r^2(N^\phi)^2}{r}\right)F_{21}^a, \end{aligned} \quad (27)$$

where the indices inside the parenthesis are algebra indices which run over 0,1,2 and each denote a different field. In this way, we have *three different electric and magnetic fields*. The radial components of three electric fields have the following forms:

$$E_r^{(0)} = -\frac{J}{r^3}h(r), \quad E_r^{(1)} = \frac{Jk}{rg(r)}, \quad E_r^{(2)} = 0, \quad (28)$$

furthermore, all of the azimuthal components of them are zero

$$E_\phi^{(0)} = E_\phi^{(1)} = E_\phi^{(2)} = 0, \quad (29)$$

and all of the magnetic fields are in z-direction i.e. they are perpendicular to r, ϕ plane. These magnetic fields are given by

$$B_z^{(0)} = -\frac{J^2}{2r^4}h(r) + \frac{N^2}{r}\frac{d}{dr}h(r), \quad B_z^{(1)} = \frac{2k(M + \Lambda r^2)}{g(r)} + \frac{krN^2}{(g(r))^2}\frac{d}{dr}g(r), \quad B_z^{(2)} = 0. \quad (30)$$

4 Dual solution

Now, we will try to find the dual of the Abelian solution (19) which is indeed another solution for the equations of motion. Note that the solution (19) is independent of the coordinate φ ; hence we have an isometry in φ direction. Abelian duality which is a transformation on the string model, relates this solution $(g_{\mu\nu}, B_{\mu\nu}, F_{\mu\nu}^a, \phi)$ to the dual solution $(\tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{F}_{\mu\nu}^a, \tilde{\phi})$ by the following Buscher's transformation [20]

$$\begin{aligned} \tilde{g}_{22} &= \frac{1}{g_{22}}, & \tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - \frac{(g_{2\alpha}g_{2\beta} - B'_{2\alpha}B'_{2\beta})}{g_{22}}, \\ \tilde{g}_{2\alpha} &= \frac{B'_{2\alpha}}{g_{22}}, & \tilde{B}'_{\alpha\beta} &= B'_{\alpha\beta} - \frac{2g_{2[\alpha}B'_{\beta]2}}{g_{22}}, \\ \tilde{B}'_{2\alpha} &= \frac{g_{2\alpha}}{g_{22}}, & \tilde{\phi} &= \phi - \frac{1}{2}\ln(g_{22}), \end{aligned} \quad (31)$$

where $\alpha, \beta = 0, 1$ (indice '2' represents the φ coordinate). Applying this transformation to the solution (11),(12),(19) and (21) yields the following dual solution:

$$d\tilde{s}^2 = \left(M - \frac{J^2}{4r^2}\right)dt^2 + \frac{2}{\ell}dtd\varphi + 2B'_{21}(r)\left(\frac{1}{\ell}dtdr + \frac{1}{r^2}drd\varphi\right) + \frac{1}{r^2}d\varphi^2 + \left\{\frac{1}{N^2(r)} + \frac{1}{r^2}\left(B'_{21}(r)\right)^2\right\}dr^2,$$

$$\begin{aligned}\tilde{B}_{20}(r) &= -\frac{J}{2r^2}, & \tilde{\phi}(r) &= -\ln(r), \\ \tilde{F}_{01}{}^0(r) &= -\frac{J}{r^3}h(r) + \frac{3J}{2r^2}\frac{d}{dr}h(r), & \tilde{F}_{01}{}^1(r) &= -\frac{2Jk}{rg(r)} + \frac{3Jk}{2g^2(r)}\frac{d}{dr}g(r),\end{aligned}\quad (32)$$

where the dual electromagnetic field strengths $\tilde{F}_{01}{}^0$ and $\tilde{F}_{01}{}^1$, are related to the following gauge fields (using (20)):

$$\begin{aligned}\tilde{A}_0{}^0(r) &= -\int dr \left(-\frac{J}{r^3}h(r) + \frac{3J}{2r^2}\frac{d}{dr}h(r)\right), \\ \tilde{A}_0{}^1(r) &= -\int dr \left(-\frac{2Jk}{rg(r)} + \frac{3Jk}{2g^2(r)}\frac{d}{dr}g(r)\right), \\ \tilde{A}_1{}^2(r) &= \frac{-1}{N^2(r)}\int dr \left(-\frac{J}{r^3}h(r) + \frac{3J}{2r^2}\frac{d}{dr}h(r)\right).\end{aligned}\quad (33)$$

Note that for $B'_{21}(r) = 0$, and using the following coordinate transformation:

$$r^2 = \ell r', \quad t = \frac{\sqrt{\ell}(\varphi' - t')}{(M^2\ell^2 - J^2)^{\frac{1}{4}}}, \quad \varphi = \frac{r_+^2 t' - r_-^2 \varphi'}{\sqrt{\ell}(M^2\ell^2 - J^2)^{\frac{1}{4}}}, \quad (34)$$

the dual metric in (32) precisely represents the charged black string solution [3, 4]. The dual electric and magnetic fields can be obtained in terms of dual field strengths as follows:

$$\begin{aligned}\tilde{E}_r^{(a)} &= -\tilde{F}^{01a} = -\tilde{g}^{00}\tilde{g}^{11}\tilde{F}_{01}{}^a, & \tilde{E}_\varphi^{(a)} &= -\tilde{F}^{02a} = -\tilde{g}^{00}\tilde{g}^{21}\tilde{F}_{01}{}^a, \\ \tilde{B}_z^{(a)} &= r\tilde{F}^{12a} = -r\tilde{g}^{11}\tilde{g}^{20}\tilde{F}_{01}{}^a = -\frac{r^3}{\ell}\tilde{F}_{01}{}^a.\end{aligned}\quad (35)$$

Furthermore, the radial components of three dual electric fields are obtained as:

$$\tilde{E}_r^{(0)} = -\frac{J}{r^3}h(r) + \frac{3J}{2r^2}\frac{d}{dr}h(r), \quad \tilde{E}_r^{(1)} = -\frac{2Jk}{rg(r)} + \frac{3Jk}{2g^2(r)}\frac{d}{dr}g(r), \quad \tilde{E}_r^{(2)} = 0, \quad (36)$$

and the azimuthal components of them are obtained as follows:

$$\begin{aligned}\tilde{E}_\varphi^{(0)} &= \left(\frac{d}{dr}h(r) - \frac{2kr}{g(r)} + \frac{kr^2}{g^2(r)}\frac{d}{dr}g(r)\right) \left(-\frac{J}{r^3}h(r) + \frac{3J}{2r^2}\frac{d}{dr}h(r)\right), \\ \tilde{E}_\varphi^{(1)} &= \left(\frac{d}{dr}h(r) - \frac{2kr}{g(r)} + \frac{kr^2}{g^2(r)}\frac{d}{dr}g(r)\right) \left(-\frac{2Jk}{rg(r)} + \frac{3Jk}{2g^2(r)}\frac{d}{dr}g(r)\right), \\ \tilde{E}_\varphi^{(2)} &= 0,\end{aligned}\quad (37)$$

and finally the magnetic fields which all are in z-direction are given by the following relations:

$$\tilde{B}_z^{(0)} = -\frac{r^3}{\ell} \left(-\frac{J}{r^3}h(r) + \frac{3J}{2r^2}\frac{d}{dr}h(r)\right), \quad \tilde{B}_z^{(1)} = -\frac{r^3}{\ell} \left(-\frac{2Jk}{rg(r)} + \frac{3Jk}{2g^2(r)}\frac{d}{dr}g(r)\right), \quad \tilde{B}_z^{(2)} = 0. \quad (38)$$

Note that if we select the following forms for arbitrary functions $g(r)$ and $h(r)$:

$$g(r) = DrN(r) e^{-V(r)}, \quad h(r) = \frac{Cr}{N(r)} e^{V(r)}, \quad (39)$$

where D and C are arbitrary constants and

$$V(r) = \frac{M}{\sqrt{M^2 + \Lambda J^2}} \tanh^{-1} \left(\frac{M + 2\Lambda r^2}{\sqrt{M^2 + \Lambda J^2}} \right), \quad (40)$$

then all magnetic fields in (30) become zero, and therefore, we have only the electric fields in solution (19), while in dual solution (32) there are both electric and magnetic fields. This result, as we expect, indicates that

here, in string theory, the duality leads to a connection between electric and magnetic fields in solution (19) and it's dual solution (32).

At the end of this section let us discuss about the isometry symmetry along t direction of the solution (19). One can repeat the above procedure to find the following t -dual solution [21] for solution (19):

$$\begin{aligned}\tilde{ds}^2 &= \frac{1}{(M - \frac{r^2}{\ell^2})} \left[dt^2 + 2\frac{r^2}{\ell} dt d\varphi - 2B'_{01}(r) \left(dt dr - \frac{r^2}{\ell} dr d\varphi \right) + r^2 \left(M - \frac{J^2}{4r^2} \right) d\varphi^2 + \left\{ \frac{(M - \frac{r^2}{\ell^2})}{N^2(r)} + \left(B'_{01}(r) \right)^2 \right\} dr^2 \right], \\ \tilde{B}_{20}(r) &= -\frac{J}{2(M - \frac{r^2}{\ell^2})}, \quad \tilde{\phi}(r) = -\frac{1}{2} \ln(-M + \frac{r^2}{\ell^2}), \\ \tilde{F}_{21}{}^0(r) &= \frac{1}{(M - \frac{r^2}{\ell^2})} \left(\frac{J^2}{r^3} h(r) + (-M + \frac{r^2}{\ell^2} - \frac{J^2}{2r^2}) \frac{d}{dr} h(r) \right), \\ \tilde{F}_{21}{}^1(r) &= \frac{2kr}{g(r)} + \frac{kr^2}{g^2(r)(M - \frac{r^2}{\ell^2})} (-M + \frac{r^2}{\ell^2} - \frac{J^2}{2r^2}) \frac{d}{dr} g(r),\end{aligned}\tag{41}$$

For $B'_{01}(r) = 0$, and using the following coordinate transformation [21]:

$$r^2 = \ell r' + M\ell^2, \quad t = \frac{r_+^2 \varphi' - r_-^2 t'}{\ell^{\frac{3}{2}} (M^2 \ell^2 - J^2)^{\frac{1}{4}}}, \quad \varphi = \frac{t' - \varphi'}{\sqrt{\ell} (M^2 \ell^2 - J^2)^{\frac{1}{4}}},\tag{42}$$

the t -dual metric in (41) precisely reduces to the charged black string solution. Finally, calculations similar to the φ -dual case show that t -dual solution also leads to a connection between electric and dual magnetic fields for solution (19) and it's t -dual solution, respectively.

5 Conclusions

We have presented a $2+1$ dimensional low energy string effective action containing gauge fields term which has a gauge symmetry coming from semi-simple extension of Poincaré (Maxwell) gauge group. The model has led to an extended B-field in the corresponding sigma model. By solving the equations of motion of the string effective action (i.e. the beta function equations), we have obtained two different solutions which both in the sigma model level correspond to the $SL(2, R)$ WZW models and then, both are exact solutions of beta function equations to all orders. Also, it turned out that two sigma models corresponding to two different solutions are classically canonically equivalent. We have interpreted the gauge field strength tensors related to the Abelian gauge fields solution as electromagnetic field strength tensors and obtained the corresponding electric and magnetic fields. Using Buscher duality transformation, we have shown that the dual models coincide with the charged black string solution and also have shown that the electric fields of the Abelian solution are related to the magnetic fields of it's dual solution.

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A Appendix: S-expansion of the anti-de Sitter (AdS) algebra $so(2, 2)$ in $2+1$ dimensions

In this section, we start from the anti-de Sitter algebra $\mathfrak{g} = so(2, 2)$ in $2+1$ dimensions and use the finite abelian semigroup expansion procedure⁶ (S-expansion) to cast both the Maxwell algebra and the semi-simple extension

⁶The full details of the S-expansion procedure have been presented in [17].

of the Poincaré algebra.⁷ We consider the anti-de Sitter algebra $\mathfrak{g} = so(2, 2)$ with the basis $\overline{X}_B = \{\overline{P}_a, \overline{J}_a\}$ in $2 + 1$ dimensions as follows:

$$[\overline{J}_a, \overline{J}_b] = \epsilon_{abc} \overline{J}^c, \quad [\overline{J}_a, \overline{P}_b] = \epsilon_{abc} \overline{P}^c, \quad [\overline{P}_a, \overline{P}_b] = \Lambda \epsilon_{abc} \overline{J}^c, \quad (43)$$

where Λ is a constant, and \overline{P}_a and \overline{J}_a are the ordinary translation and Lorentz generators, respectively. We split the *Ads* algebra $so(2, 2) = V_0 \oplus V_1$, where V_0 and V_1 corresponds to the Lorentz and translation generators \overline{J}_a and \overline{P}_a , respectively. The subspace structure is such that we have:

$$[V_0, V_0] \subset V_0, \quad [V_0, V_1] \subset V_1, \quad [V_1, V_1] \subset V_0. \quad (44)$$

Now, in the following, we use two different semigroups to expand the *Ads* algebra by use of the S-expansion procedure, and obtain both the Maxwell algebra and the semi-simple extension of the Poincaré algebra.

A.1 The Maxwell algebra and the S-expansion by the semigroup S

We first consider the abelian semigroup $S = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ together with the following multiplication law:

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_3 & \text{if } (\alpha + \beta) > 2 \\ \lambda_{\alpha+\beta} & \text{if } (\alpha + \beta) \leq 2 \end{cases}, \quad (45)$$

or, equivalently, by the following multiplication table:

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_3
λ_2	λ_2	λ_3	λ_3	λ_3
λ_3	λ_3	λ_3	λ_3	λ_3

(46)

Note that for each $\lambda_\alpha \in S$, we have $\lambda_3 \lambda_\alpha = \lambda_3$, such that λ_3 plays the role of the zero element inside the semigroup S (i.e. $\lambda_3 = 0_S$). Now, we consider the following subset decomposition $S = S_0 \cup S_1$:

$$S_0 = \{\lambda_0, \lambda_2, \lambda_3\}, \quad S_1 = \{\lambda_1, \lambda_3\}, \quad (47)$$

which is said to be a resonant decomposition, in other words, it is in resonance with the subspace decomposition $\mathfrak{g} = V_0 \oplus V_1$ and then, satisfies the following resonance condition:

$$S_0 \cdot S_0 \subset S_0, \quad S_0 \cdot S_1 \subset S_1, \quad S_1 \cdot S_1 \subset S_0. \quad (48)$$

The direct product $S \times \mathfrak{g}$ with basis $\lambda_\alpha \overline{X}_B$ is a Lie algebra (see Theorem III.1 in [17]). According to the Theorem IV.2 in [17], $W_0 \oplus W_1$ is a resonant subalgebra of $S \times \mathfrak{g}$ where we have:

$$W_0 = S_0 \times V_0 = \{\lambda_0, \lambda_2, \lambda_3\} \otimes \{\overline{J}_a\} = \{\lambda_0 \overline{J}_a, \lambda_2 \overline{J}_a, \lambda_3 \overline{J}_a\}, \quad (49)$$

$$W_1 = S_1 \times V_1 = \{\lambda_1, \lambda_3\} \otimes \{\overline{P}_a\} = \{\lambda_1 \overline{P}_a, \lambda_3 \overline{P}_a\}. \quad (50)$$

Now, we impose the condition $\lambda_3 \times \mathfrak{g} = 0_S$, and remove the whole $0_S \times \mathfrak{g}$ sector from the resonant subalgebra. The remaining piece is a Lie algebra and is called the 0_S -reduced algebra (see 0_S -reduction and Theorem VI.1 in [17]). By relabeling the generators as $J_a \equiv \lambda_0 \overline{J}_a$, $kZ_a \equiv \lambda_2 \overline{J}_a$ and $\sqrt{\Lambda} P_a \equiv \lambda_1 \overline{P}_a$, we obtain the following commutation relations:

$$\begin{aligned} [J_a, J_b] &= \lambda_0 \lambda_0 [\overline{J}_a, \overline{J}_b] = \lambda_0 \epsilon_{abc} \overline{J}^c = \epsilon_{abc} J^c, \\ [J_a, P_b] &= \frac{1}{\sqrt{\Lambda}} \lambda_0 \lambda_1 [\overline{J}_a, \overline{P}_b] = \frac{1}{\sqrt{\Lambda}} \lambda_1 \epsilon_{abc} \overline{P}^c = \epsilon_{abc} P^c, \\ [P_a, P_b] &= \frac{1}{\Lambda} \lambda_1 \lambda_1 [\overline{P}_a, \overline{P}_b] = \lambda_2 \epsilon_{abc} \overline{J}^c = k \epsilon_{abc} Z^c, \end{aligned}$$

⁷The Maxwell algebra and the semi-simple extension of the Poincaré algebra in D -dimensional spacetime have been obtained by the S-expansion of the D -dimensional *Ads* algebra in [13] and [14], respectively.

$$\begin{aligned}
[J_a, Z_b] &= \frac{1}{k} \lambda_0 \lambda_2 [\bar{J}_a, \bar{J}_b] = \frac{1}{k} \lambda_2 \epsilon_{abc} \bar{J}^c = \epsilon_{abc} Z^c, \\
[P_a, Z_b] &= \frac{1}{k\sqrt{\Lambda}} \lambda_1 \lambda_2 [\bar{P}_a, \bar{J}_b] = \frac{1}{k\sqrt{\Lambda}} \epsilon_{abc} \lambda_3 \bar{P}^c = 0, \\
[Z_a, Z_b] &= \frac{1}{k^2} \lambda_2 \lambda_2 [\bar{J}_a, \bar{J}_b] = \frac{1}{k^2} \epsilon_{abc} \lambda_3 \bar{J}^c = 0,
\end{aligned} \tag{51}$$

where we have used the commutation relations of the *Ads* algebra (43) and the multiplication law (46) of the semigroup S . The obtained algebra (51) coincides with the Maxwell algebra [8, 9].

A.2 The semi-simple extension of the Poincaré algebra and the S-expansion by the semigroup \bar{S}

We consider the abelian semigroup $\bar{S} = \{\bar{\lambda}_0, \bar{\lambda}_1, \bar{\lambda}_2\}$ together with the following multiplication law:

$$\bar{\lambda}_\alpha \bar{\lambda}_\beta = \begin{cases} \bar{\lambda}_{\alpha+\beta-2} & \text{if } (\alpha + \beta) > 2 \\ \bar{\lambda}_{\alpha+\beta} & \text{if } (\alpha + \beta) \leq 2 \end{cases}, \tag{52}$$

or, equivalently, by the following multiplication table:

	$\bar{\lambda}_0$	$\bar{\lambda}_1$	$\bar{\lambda}_2$
$\bar{\lambda}_0$	$\bar{\lambda}_0$	$\bar{\lambda}_1$	$\bar{\lambda}_2$
$\bar{\lambda}_1$	$\bar{\lambda}_1$	$\bar{\lambda}_2$	$\bar{\lambda}_1$
$\bar{\lambda}_2$	$\bar{\lambda}_2$	$\bar{\lambda}_1$	$\bar{\lambda}_2$

(53)

Now, we consider the following subset decomposition $\bar{S} = \bar{S}_0 \cup \bar{S}_1$:

$$\bar{S}_0 = \{\bar{\lambda}_0, \bar{\lambda}_2\}, \quad \bar{S}_1 = \{\bar{\lambda}_1\}, \tag{54}$$

which is in resonance with the subspace decomposition $\mathfrak{g} = V_0 \oplus V_1$ (resonant decomposition) and then, satisfies the following resonance condition:

$$\bar{S}_0 \cdot \bar{S}_0 \subset \bar{S}_0, \quad \bar{S}_0 \cdot \bar{S}_1 \subset \bar{S}_1, \quad \bar{S}_1 \cdot \bar{S}_1 \subset \bar{S}_0. \tag{55}$$

The direct product $\bar{S} \times \mathfrak{g}$ with basis $\bar{\lambda}_\alpha \bar{X}_B$ is a Lie algebra (see Theorem III.1 in [17]). According to the Theorem IV.2 in [17], $\bar{W}_0 \oplus \bar{W}_1$ is a resonant subalgebra of $\bar{S} \times \mathfrak{g}$ where we have:

$$\begin{aligned}
\bar{W}_0 &= \bar{S}_0 \times V_0 = \{\bar{\lambda}_0, \bar{\lambda}_2\} \otimes \{\bar{J}_a\} = \{\bar{\lambda}_0 \bar{J}_a, \bar{\lambda}_2 \bar{J}_a\}, \\
\bar{W}_1 &= \bar{S}_1 \times V_1 = \{\bar{\lambda}_1\} \otimes \{\bar{P}_a\} = \{\bar{\lambda}_1 \bar{P}_a\}.
\end{aligned} \tag{56}$$

Relabeling the generators as $J_a \equiv \bar{\lambda}_0 \bar{J}_a$, $-\frac{k}{\lambda} Z_a \equiv \bar{\lambda}_2 \bar{J}_a$ and $\sqrt{-\frac{\Lambda}{\lambda}} P_a \equiv \bar{\lambda}_1 \bar{P}_a$, we obtain the following commutation relations:

$$\begin{aligned}
[J_a, J_b] &= \bar{\lambda}_0 \bar{\lambda}_0 [\bar{J}_a, \bar{J}_b] = \bar{\lambda}_0 \epsilon_{abc} \bar{J}^c = \epsilon_{abc} J^c, \\
[J_a, P_b] &= \sqrt{-\frac{\lambda}{\Lambda}} \bar{\lambda}_0 \bar{\lambda}_1 [\bar{J}_a, \bar{P}_b] = \sqrt{-\frac{\lambda}{\Lambda}} \bar{\lambda}_1 \epsilon_{abc} \bar{P}^c = \epsilon_{abc} P^c, \\
[P_a, P_b] &= -\frac{\lambda}{\Lambda} \bar{\lambda}_1 \bar{\lambda}_1 [\bar{P}_a, \bar{P}_b] = -\lambda \bar{\lambda}_2 \epsilon_{abc} \bar{J}^c = k \epsilon_{abc} Z^c, \\
[J_a, Z_b] &= -\frac{\lambda}{k} \bar{\lambda}_0 \bar{\lambda}_2 [\bar{J}_a, \bar{J}_b] = -\frac{\lambda}{k} \bar{\lambda}_2 \epsilon_{abc} \bar{J}^c = \epsilon_{abc} Z^c, \\
[P_a, Z_b] &= -\frac{\lambda}{k} \sqrt{-\frac{\lambda}{\Lambda}} \bar{\lambda}_1 \bar{\lambda}_2 [\bar{P}_a, \bar{J}_b] = -\frac{\lambda}{k} \sqrt{-\frac{\lambda}{\Lambda}} \bar{\lambda}_1 \epsilon_{abc} \bar{P}^c = -\frac{\lambda}{k} \epsilon_{abc} \bar{P}^c, \\
[Z_a, Z_b] &= \frac{\lambda^2}{k^2} \bar{\lambda}_2 \bar{\lambda}_2 [\bar{J}_a, \bar{J}_b] = \frac{\lambda^2}{k^2} \bar{\lambda}_2 \epsilon_{abc} \bar{J}^c = -\frac{\lambda}{k} \epsilon_{abc} \bar{Z}^c,
\end{aligned} \tag{57}$$

where we have used the commutation relations of the *Ads* algebra (43) and the multiplication law (53) of the semigroup \bar{S} . The obtained algebra (57) matches the semi-simple extension of the Poincaré algebra [11].

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